Rationalizability, Cost-Rationalizability, and Afriat's Efficiency Index

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1 Introduction

The notion of *rationalizability* by utility maximization is natural, standard, and normatively uncontroversial in the revealed preference literature—a utility function is said to *rationalize* observed expenditure data if, at each observation, it assigns weakly higher utility to the chosen bundle than to any other bundle that is weakly cheaper at the prevailing prices. An equally natural dual notion, but one which is much less explicitly mentioned in the literature (even though it implicitly justifies many empirical studies of consumer demand), is *cost-rationalizability*, where a given utility target is achieved with minimum expenditure. In this note, we show under what (mild) conditions, the two notions of rationalizabilty are *definitionally equivalent*; we also show that they are *observationally equivalent* in the context of a data set with finitely many observations. In most empirical settings, a data set will not be exactly rationalizable or cost-rationalizable and some way of measuring its departure from exact rationalizability is needed. The most common way to do this is via the *critical cost efficiency index* (CCEI) of Afriat (1973), or its closely related generalization by Varian (1990). We explain how Afriat's efficiency index can be interpreted within the context of cost-rationalizability.

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2 Approximate Rationalizability

At observation t, the prices of ℓ goods are $\mathbf{p}^t$ and the consumer purchases the bundle $\mathbf{x}^t$ of these ℓ goods. We refer to the collection of observations, $\mathcal{O} = \{\mathbf{p}^t, \mathbf{x}^t\}_{t=1}^T$, as a *data set*. We assume throughout that $\mathbf{p}^t \in \mathbb{R}^\ell_{++}$ and $\mathbf{x}^t \in \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$.

Definition 1. Let $\mathcal{U}$ be a collection of utility functions defined on the consumption space $\mathbb{R}^\ell_+$. The data set $\mathcal{O} = \{\mathbf{p}^t, \mathbf{x}^t\}_{t=1}^T$ is *rationalizable in $\mathcal{U}$ with respect to the vector of efficiency coefficients* $\mathbf{e} = \{e^t\}_{t=1}^T$ (where $e^t \in (0, 1]$ for all t) (or simply $\mathbf{e}$-rationalizable in $\mathcal{U}$) if there is a utility function $U : \mathbb{R}^\ell_+ \to \mathbb{R}$ belonging to $\mathcal{U}$, such that, at every observation t,

$$U(\mathbf{x}^t) \geqslant U(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathcal{B}^t(e^t) = \{\mathbf{x}' \in \mathbb{R}^\ell_+ : \mathbf{p}^t \cdot \mathbf{x}' \leqslant e^t \mathbf{p}^t \cdot \mathbf{x}^t\}.$$

A data set $\mathcal{O}$ is said to be *exactly* rationalizable (or simply *rationalizable*) in $\mathcal{U}$ if it is $(1, 1, \ldots, 1)$-rationalizable in $\mathcal{U}$.

Loosely speaking, the further is $\mathbf{e}$ from $(1, 1, \ldots, 1)$, the weaker is the rationalizability displayed within the data set. Afriat (1973) proposes the *critical cost efficiency index* (CCEI) as a measure of a data set's distance from exact rationalizability.

Definition 2. Given a data set $\mathcal{O}$ and a family of utility functions $\mathcal{U}$, the CCEI is given by

$$\sup\{e : \mathcal{O} \text{ is } (e, e, \ldots, e)\text{-rationalizable in } \mathcal{U}\}. \tag{1}$$

To understand what the CCEI means, suppose that a data set $\mathcal{O}$ has a CCEI of 0.95 for the family of utility functions $\mathcal{U}$. Then the agent's observed choices are not exactly rationalizable in the sense that for *any* utility function U in $\mathcal{U}$, there is some observation s and bundle $\mathbf{x}'$ such that $U(\mathbf{x}^s) < U(\mathbf{x}')$ and $\mathbf{p}^s \cdot \mathbf{x}' \leqslant \mathbf{p}^s \cdot \mathbf{x}^s$. However, this 'irrationality' is limited in the sense that for any $\varepsilon > 0$, there is a utility function $\bar{U}$ in $\mathcal{U}$ such that $\bar{U}(\mathbf{x}^t) \geqslant \bar{U}(\mathbf{x})$ for any $\mathbf{x}$ satisfying $(0.95 - \varepsilon)\mathbf{p}^t \cdot \mathbf{x}^t \geqslant \mathbf{p}^t \cdot \mathbf{x}$ for all observations t; in other words, $\mathbf{x}^t$ gives weakly higher utility than any bundle that is more than 5% cheaper than $\mathbf{x}^t$.

Afriat (1973) provides a way to calculate the CCEI. To be precise, it is shown that if $\mathcal{O}$ is $\mathbf{e}$-rationalizable by a member of $\mathcal{U}_{LNS}$, the family of *locally nonsatiated* utility functions, then $\mathcal{O}$ must obey a particular property that is easy to check; furthermore, if a data set satisfies this property then it is $\mathbf{e}$-rationalizable by a member of $\mathcal{U}_{WB}$, the family of *well-behaved* utility functions, in the sense that the utility functions are strictly increasing and continuous. Given this result, a data set's CCEI can be obtained via a binary search over $\mathbf{e} = (e, e, \ldots, e)$, for $e \in (0, 1]$, and the CCEI for the family $\mathcal{U}_{LNS}$ must coincide with that for the family $\mathcal{U}_{WB}$.[1]

We now introduce the concepts needed to explain Afriat's result.

DEFINITION 3. Let $\mathcal{O} = \{\mathbf{p}^t, \mathbf{x}^t\}_{t=1}^T$ be a data set and $\mathbf{e} = \{e^t\}_{t=1}^T$ a vector of efficiency coefficients. For any pair of bundles $(\mathbf{x}^t, \mathbf{x}^s)$, the bundle $\mathbf{x}^t$ is *(strictly) directly revealed preferred* to the bundle $\mathbf{x}^s$, denoted $\mathbf{x}^t \succcurlyeq_0^* (\succ_0^*)\, \mathbf{x}^s$, whenever $\mathbf{p}^t \cdot \mathbf{x}^s \leqslant (<)\, e^t \mathbf{p}^t \cdot \mathbf{x}^t$. For any finite sequence of bundles $(\mathbf{x}^t, \mathbf{x}^i, \mathbf{x}^j, \ldots, \mathbf{x}^l, \mathbf{x}^s)$, the bundle $\mathbf{x}^t$ is *revealed preferred* to the bundle $\mathbf{x}^s$, denoted $\mathbf{x}^t \succcurlyeq^* \mathbf{x}^s$, whenever $\mathbf{x}^t \succcurlyeq_0^* \mathbf{x}^i$, $\mathbf{x}^i \succcurlyeq_0^* \mathbf{x}^j$, $\ldots$, $\mathbf{x}^l \succcurlyeq_0^* \mathbf{x}^s$.

DEFINITION 4. Given a vector of efficiency coefficients $\mathbf{e} = \{e^t\}_{t=1}^T$, a data set $\mathcal{O} = \{\mathbf{p}^t, \mathbf{x}^t\}_{t=1}^T$ obeys $\mathbf{e}$-GARP (where GARP stands for the *generalized axiom of revealed preference*) so long as $\mathbf{x}^t \succcurlyeq^* \mathbf{x}^s \implies \mathbf{x}^s \not\succ_0^* \mathbf{x}^t$. $\mathcal{O}$ satisfies GARP if it satisfies $\mathbf{e}$-GARP for $\mathbf{e} = (1, 1, \ldots, 1)$.[2]

The following result is well known. For proofs, see Afriat (1973) and Halevy, Persitz, and Zrill (2018). We provide a further proof in the Appendix for the sake of completeness.

THEOREM 1. *Let $\mathcal{O} = \{\mathbf{p}^t, \mathbf{x}^t\}_{t=1}^T$ be a data set and $\mathbf{e} = \{e^t\}_{t=1}^T$ a vector of efficiency coefficients. The following statements are equivalent:*

(1) $\mathcal{O}$ is $\mathbf{e}$-rationalizable in $\mathcal{U}_{LNS}$.

(2) $\mathcal{O}$ obeys $\mathbf{e}$-GARP.

(3) There is a set of numbers $\{\phi^t, \lambda^t\}_{t=1}^T$ (with $\phi^t \in \mathbb{R}$ and $\lambda^t \in \mathbb{R}_{++}$), such that, at all t, s,

$$\phi^s \leqslant \phi^t + \lambda^t \mathbf{p}^t \cdot (\mathbf{x}^s - e^t \mathbf{x}^t).$$

[1]It is also possible to calculate the CCEI for many other families of utility functions $\mathcal{U}$; see Polisson, Quah, and Renou (2020) for characterizations of $\mathbf{e}$-rationalizability within the context of choice under risk by a member of $\mathcal{U}_{FOSD}$ (the family of well-behaved utility functions which respect first-order stochastic dominance (FOSD)) or $\mathcal{U}_{EUT}$ (the family of well-behaved utility functions which comply with expected utility theory (EUT)), among others.

[2]The term GARP is originally from Varian (1982); Afriat (1967) introduces and refers to its equivalent as *cyclical consistency*, and Afriat (1973) develops a modified version of this property (which is equivalent to $\mathbf{e}$-GARP with $\mathbf{e} = (e, e, \ldots, e)$).

(4) $\mathcal{O}$ is **e***-rationalizable in* $\mathcal{U}_{WB}$.[3]

An immediate corollary of this result is that a data set's CCEI for the family of utility functions $\mathcal{U}_{LNS}$ coincides with that for the family $\mathcal{U}_{WB}$ and is given by

$$\sup\{e : \mathcal{O} \text{ satisfies } (e, e, \ldots, e)\text{-GARP}\}. \tag{2}$$

3 Approximate Cost-Rationalizability

Definition 5. The data set $\mathcal{O} = \{\mathbf{p}^t, \mathbf{x}^t\}_{t=1}^T$ is *cost-rationalizable with respect to the vector of efficiency coefficients* $\mathbf{e} = \{e^t\}_{t=1}^T$ (or simply $\mathbf{e}$-cost-rationalizable) if there is a utility function $U : \mathbb{R}_+^\ell \to \mathbb{R}$, such that, at every observation t,

$$e^t \mathbf{p}^t \cdot \mathbf{x}^t \leqslant \mathbf{p}^t \cdot \mathbf{x} \text{ for all } \mathbf{x} \in \mathcal{P}^t = \{\mathbf{x}' \in \mathbb{R}_+^\ell : U(\mathbf{x}') \geqslant U(\mathbf{x}^t)\}.$$

A data set $\mathcal{O}$ is said to be *exactly* cost-rationalizable (or simple *cost-rationalizable*) if it is $(1, 1, \ldots, 1)$-cost-rationalizable.

Exact cost-rationalizability means that there is some utility function U for which the bundle chosen by the agent at each observation t could be understood as the cheapest way of achieving a utility target $U(\mathbf{x}^t)$ given the prevailing prices at t.

The next proposition relates rationalizability and cost-rationalizability.

Proposition 1. (a) *If a locally nonsatiated utility function U* **e***-rationalizes $\mathcal{O}$, then U also* **e***-cost-rationalizes $\mathcal{O}$.* (b) *If a continuous utility function U* **e***-cost-rationalizes $\mathcal{O}$, then U also* **e***-rationalizes $\mathcal{O}$.*

Proof. If (a) is false then there is an observation t and a bundle $\mathbf{y}$ such that $U(\mathbf{y}) \geqslant U(\mathbf{x}^t)$ with $e^t \mathbf{p}^t \cdot \mathbf{x}^t > \mathbf{p}^t \cdot \mathbf{y}$. Since U is locally nonsatiated, there is $\mathbf{y}'$ such that $U(\mathbf{y}') > U(\mathbf{y}) \geqslant U(\mathbf{x}^t)$, with $e^t \mathbf{p}^t \cdot \mathbf{x}^t > \mathbf{p}^t \cdot \mathbf{y}'$, which contradicts the $\mathbf{e}$-rationalizability of $\mathcal{O}$ by U.

For (b), we again prove by contradiction. Suppose that U $\mathbf{e}$-cost-rationalizes $\mathcal{O}$ but that there is an observation t and a bundle $\mathbf{y}$ such that $U(\mathbf{y}) > U(\mathbf{x}^t)$ with $\mathbf{p}^t \cdot \mathbf{y} \leqslant e^t \mathbf{p}^t \cdot \mathbf{x}^t$. By

[3] In fact, statement (4) could be strengthened to say that $\mathcal{O}$ is $\mathbf{e}$-rationalizable in the family of well-behaved *and concave* utility functions.

the continuity of U and the fact that $e^t\mathbf{p}^t \cdot \mathbf{x}^t > 0$, there is $\mathbf{y}'$ such $\mathbf{p}^t \cdot \mathbf{y}' < e^t\mathbf{p}^t \cdot \mathbf{x}^t$ and $U(\mathbf{y}') > U(\mathbf{x}^t)$, but this is impossible since U $\mathbf{e}$-cost-rationalizes $\mathcal{O}$. □

We know (from Theorem 1) that $\mathbf{e}$-GARP is sufficient for $\mathbf{e}$-rationalizability by a well-behaved utility function and therefore (by Proposition 1a) it is also sufficient for $\mathbf{e}$-cost-rationalizability by a utility function in that family. The necessity of $\mathbf{e}$-GARP is stated formally below.

Proposition 2. *Suppose that $\mathcal{O}$ is $\mathbf{e}$-cost-rationalizable by a continuous utility function U. Then $\mathcal{O}$ obeys $\mathbf{e}$-GARP.*

Proof. We claim that if $\mathbf{p}^t \cdot \mathbf{x}^s = e^t\mathbf{p}^t \cdot \mathbf{x}^t$, then $U(\mathbf{x}^s) \leqslant U(\mathbf{x}^t)$. Indeed, if $U(\mathbf{x}^s) > U(\mathbf{x}^t)$, then by the continuity of U, there is $\mathbf{y}$ such that $\mathbf{p}^t \cdot \mathbf{y} < e^t\mathbf{p}^t \cdot \mathbf{x}^t$, with $U(\mathbf{y}) > U(\mathbf{x}^t)$, which contradicts the $\mathbf{e}$-cost-rationalizability of $\mathcal{O}$ by U. Also if $\mathbf{p}^t \cdot \mathbf{x}^s < e^t\mathbf{p}^t \cdot \mathbf{x}^t$, then $U(\mathbf{x}^s) < U(\mathbf{x}^t)$ since U $\mathbf{e}$-cost-rationalizes $\mathcal{O}$. GARP then follows by the usual argument.[4] □

The upshot of this proposition is that, for the family of well-behaved utility functions $\mathcal{U}_{WB}$, $\mathbf{e}$-rationalizability and $\mathbf{e}$-cost-rationalizability are observationally equivalent because they are both equivalent to $\mathbf{e}$-GARP. Similarly, the CCEI (for $\mathcal{U} = \mathcal{U}_{WB}$) as defined by (1), which is equal to (2), is also equal to

$$\sup\{e : \mathcal{O} \text{ is } (e, e, \ldots, e)\text{-cost-rationalizable in } \mathcal{U}\}. \tag{3}$$

This identity gives us a different way of interpreting the CCEI beyond its definition.

4 Cost-Rationalizability and the CCEI

First, to understand *why* cost-rationalizability makes sense as a rationality criterion, imagine that we have **observational data** on a consumer's purchases of ℓ goods, which are just a fraction of all of the types of goods that she consumes. For example, we could think of $\mathbf{x}^t$ as purchases of food products from a grocery store. The agent is cost-rationalizable if there is a utility function U defined on those ℓ goods (properly speaking, a *sub*-utility function,

[4]This argument is shown in the proof of Afriat's (1973) Theorem in the Appendix.

since her purchases will cover many other (unobserved) goods) so that, at each observation t, the bundle $\mathbf{x}^t$ is the cheapest way of achieving utility level $U(\mathbf{x}^t)$. We could think of there being a background utility function covering many more goods than the ℓ goods observed; a reasonable benchmark of rational behavior would involve the consumer being cost efficient in her purchases of the ℓ goods, in order to maximize the money available for purchases of the non-ℓ goods.

It is worth emphasizing that in the definition of cost-rationalizability, the target utility is contingent on the observation t (or, if we interpret t as time, contingent on time t). The definition does *not* require that there is not a cheaper way of achieving the same *set* of utility levels, possibly after altering the time at which a particular utility level is obtained. For example, consider a data set where $\mathbf{p}^1 = (1,1)$ and $\mathbf{x}^1 = (1,1)$ at observation 1 and $\mathbf{p}^2 = (2,2)$ and $\mathbf{x}^2 = (2,2)$ at observation 2. This data set is rationalizable and cost-rationalizable (because it obeys GARP), but clearly the agent could save money if he had bought the bundle $(2,2)$ when the price was $(1,1)$ and the bundle $(1,1)$ when the price was $(2,2)$. This does not contradict cost-rationalizability because cost-rationalizability allows the agent to choose the timing of the utility targets. This greater permissiveness makes sense: observation 1 could be on food spending in normal times and observation 2 on food spending during a festive period; cost-rationalizability does not preclude the possibility of the agent choosing to derive higher utility from food during a festive period (even when food prices are higher) and lower utility from food in normal times (even when food prices are lower).

In the context of cost-rationalizabilty, what does it mean for a consumer to have a CCEI of (say) 0.9? This means that for any $e' = 0.9 - \varepsilon$, $\mathcal{O}$ is cost-rationalizable with respect to $(e', e', \ldots, e')$. Then there is a well-behaved utility function U defined on the ℓ goods, such that there is no observation t at which the agent could have saved strictly more than $100(0.1 + \varepsilon)\%$ of her observed expenditure $\mathbf{p}^t \cdot \mathbf{x}^t$ while achieving utility $U(\mathbf{x}^t)$. In other words, there could be one (or more) observation(s) at which she is spending more than is necessary given her utility target, but the cost saving does not exceed 10% at each observation. Furthermore, given that 0.9 is the supremum among all possible values of e at which $\mathcal{O}$ is $(e, e, \ldots, e)$-rationalizable, for *any* well-behaved utility function U, there must be at least one observation $\hat{t}$ such that the agent could have achieved utility $U(\mathbf{x}^{\hat{t}})$ and saved strictly more than $100(0.1 - \varepsilon)\%$ of her observed expenditure $\mathbf{p}^{\hat{t}} \cdot \mathbf{x}^{\hat{t}}$.

In the context of **laboratory data**, suppose that a subject's data set is exactly rationalizable. Then the data set is also exactly cost-rationalizable, which means there is a utility function for which the subject will be strictly worse off should the experimenter give the subject a smaller budget (less of the experimental currency) in any one round. The CCEI is a measure of how close the subject is to exact cost-rationalizability. A CCEI of 0.9 means that, for a subject with *any* well-behaved utility function, there will be one round (and possibly more) in which the subject could have achieved the same utility in that round with a budget that is $100(0.1 - \varepsilon)\%$ smaller. However, for any withdrawal of $100(0.1 + \varepsilon)\%$ of the budget, there is a well-behaved utility function such that a subject with such a utility function is strictly worse off in some round.

Appendix

Proof of Theorem 1. (1) $\implies$ (2): The $\mathbf{e}$-rationalizability of $\mathcal{O}$ by U in $\mathcal{U}_{LNS}$ guarantees that $\mathbf{p}^t \cdot \mathbf{x}^s \leqslant e^t \mathbf{p}^t \cdot \mathbf{x}^t \implies U(\mathbf{x}^t) \geqslant U(\mathbf{x}^s)$ and $\mathbf{p}^t \cdot \mathbf{x}^s < e^t \mathbf{p}^t \cdot \mathbf{x}^t \implies U(\mathbf{x}^t) > U(\mathbf{x}^s)$. The first implication follows directly from the definition of $\mathbf{e}$-rationalizability of $\mathcal{O}$ by U, and the second from the definition of $\mathbf{e}$-rationalizability plus the local nonsatiation of U.[5] Therefore, for any $\mathbf{p}^t \cdot \mathbf{x}^i \leqslant e^t \mathbf{p}^t \cdot \mathbf{x}^t$, $\mathbf{p}^i \cdot \mathbf{x}^j \leqslant e^i \mathbf{p}^i \cdot \mathbf{x}^i$, ... , $\mathbf{p}^l \cdot \mathbf{x}^t \leqslant e^l \mathbf{p}^l \cdot \mathbf{x}^l$, we obtain

$$U(\mathbf{x}^t) \geqslant U(\mathbf{x}^i) \geqslant U(\mathbf{x}^j) \geqslant \cdots \geqslant U(\mathbf{x}^l) \geqslant U(\mathbf{x}^t),$$

which of course implies that $U(\mathbf{x}^t) = U(\mathbf{x}^i) = U(\mathbf{x}^j) = \cdots = U(\mathbf{x}^l) = U(\mathbf{x}^t)$. This equality then equires that $\mathbf{p}^t \cdot \mathbf{x}^i = e^t \mathbf{p}^t \cdot \mathbf{x}^t$, $\mathbf{p}^i \cdot \mathbf{x}^j = e^i \mathbf{p}^i \cdot \mathbf{x}^i$, ... , $\mathbf{p}^l \cdot \mathbf{x}^t = e^l \mathbf{p}^l \cdot \mathbf{x}^l$. In other words, for any revealed preference cycle $\mathbf{x}^t \succcurlyeq_0^* \mathbf{x}^i$, $\mathbf{x}^i \succcurlyeq_0^* \mathbf{x}^j$, ... , $\mathbf{x}^l \succcurlyeq_0^* \mathbf{x}^t$, the weak relation $\succcurlyeq_0^*$ cannot be replaced with the strict relation $\succ_0^*$ anywhere in the cycle.

(2) $\implies$ (3): Define $a^{t,s} = \mathbf{p}^t \cdot (\mathbf{x}^s - e^t \mathbf{x}^t)$ for all t, s, and let $\mathbf{A}$ be a $T \times T$ square matrix with $a^{t,s}$ the (t, s)-th entry. Since $\mathcal{O}$ satisfies $\mathbf{e}$-GARP, $\mathbf{A}$ satisfies cyclical consistency in the sense defined in Forges and Minelli (2009), which guarantees that there is a set of numbers

[5] To see the latter, suppose that $\mathbf{p}^t \cdot \mathbf{x}^s < e^t \mathbf{p}^t \cdot \mathbf{x}^t \implies U(\mathbf{x}^t) = U(\mathbf{x}^s)$. (From the $\mathbf{e}$-rationalizability of $\mathcal{O}$ by U, $\mathbf{p}^t \cdot \mathbf{x}^s < e^t \mathbf{p}^t \cdot \mathbf{x}^t \implies U(\mathbf{x}^t) \geqslant U(\mathbf{x}^s)$.) Then by the local nonsatiation of U, there must be some bundle $\mathbf{y}$ such that $\mathbf{p}^t \cdot \mathbf{y} < e^t \mathbf{p}^t \cdot \mathbf{x}^t$ and $U(\mathbf{y}) > U(\mathbf{x}^s) = U(\mathbf{x}^t)$, contradicting $\mathbf{e}$-rationalizability.

$\{\phi^t, \lambda^t\}_{t=1}^T$ (with $\phi^t \in \mathbb{R}$ and $\lambda^t \in \mathbb{R}_{++}$), such that, at all t, s, $\phi^s \leqslant \phi^t + \lambda^t a^{t,s}$.

(3) $\implies$ (4): The construction argument used here follows Afriat (1973). For any $\mathbf{x}$ define $U(\mathbf{x}) = \min_t\{\phi^t + \lambda^t \mathbf{p}^t \cdot (\mathbf{x} - e^t \mathbf{x}^t)\}$, and notice that U is strictly increasing, continuous, and concave. First, we must have $U(\mathbf{x}^t) \geqslant \phi^t$. To see this, suppose that $U(\mathbf{x}^t) < \phi^t$; then $U(\mathbf{x}^t) = \phi^m + \lambda^m \mathbf{p}^m \cdot (\mathbf{x}^t - e^m \mathbf{x}^m) < \phi^t$, contradicting (3). Second, for any $\mathbf{x}$ satisfying $\mathbf{p}^t \cdot \mathbf{x} \leqslant e^t \mathbf{p}^t \cdot \mathbf{x}^t$, we have $\phi^t \geqslant \phi^t + \lambda^t \mathbf{p}^t \cdot (\mathbf{x} - e^t \mathbf{x}^t)$ since $\lambda^t > 0$, and $U(\mathbf{x}) \leqslant \phi^t + \lambda^t \mathbf{p}^t \cdot (\mathbf{x} - e^t \mathbf{x}^t)$ by the definition of U. Taken altogether,

$$U(\mathbf{x}^t) \geqslant \phi^t \geqslant \phi^t + \lambda^t \mathbf{p}^t \cdot (\mathbf{x} - e^t \mathbf{x}^t) \geqslant U(\mathbf{x}).$$

(4) $\implies$ (1): This is obvious since every well-behaved utility function is (by definition) strictly increasing and thus locally nonsatiated. □

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