A Lattice Test for Additive Separability

Matthew Polisson

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University of St Andrews

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How Restrictive Is Additive Separability?

"... the main argument of this paper is that the assumption of additive preferences is almost certain to be invalid in practice and the use of demand models based on such an assumption will lead to severe distortion of measurement. So that if the price to be paid for the theoretical consistency of demand models is the necessity of assuming additive preferences, then the price is too high."

—Angus Deaton, *Economic Journal*, 1974

A Revealed Preference Test for Additive Separability

The *separability* of an agent's preference (across goods, time periods, and states of the world) is a frequently held assumption in economics.

Separability is very useful because it *restricts* the attention of both the agent and the econometrician to a single commodity or group.

- While this restriction pays dividends in numerous theoretical and empirical dimensions, it does not come without a cost.

One might therefore want to *test* for the separability of a consumer's preference, i.e., to characterize the empirical content of separability in the form of its implied restrictions on observable data.

We *develop* and *implement* revealed preference tests for a strong but common form of separability, known as *additive separability*.

What Is Separability?

Suppose that a consumption set can be divided into two commodity groups X and Y, i.e., the consumption set is given by $X \times Y$.

A preference relation $\succcurlyeq$ is said to be *separable* on X if $(x, y) \succcurlyeq (x', y)$ for some $y \in Y$ implies $(x, y') \succcurlyeq (x', y')$ for all $y' \in Y$.

If $\succcurlyeq$ admits a utility representation, then preferences are separable on X if and only if $u(x, y) = f(v(x), y)$, where f is increasing in v.

The separability of a consumer's preference is often an assumption which is made out of necessity, either explicitly or implicitly.

- Empirical work in consumer demand and industrial organization necessarily focuses on a subset of products.

Substitution patterns across groups are necessarily restricted *a priori*, e.g., across goods, time periods, and states of the world.

What Is Separability?

Many canonical models appeal to even *stronger* forms of separability, e.g., quasilinear utility, exponential discounting, and expected utility.

The strongest form of separability typically invoked is *additive* across all commodities, i.e., if there are ℓ commodities, then

$$u(x_1, x_2, \ldots, x_\ell) = \sum_{i=1}^{\ell} v_i(x_i),$$

where $v_i : \mathbb{R}_+ \to \mathbb{R}$ for all $i = 1, 2, \ldots, \ell$.

Furthermore, Debreu (1960) has shown that the weak separability of every subset implies *additive* separability, and vice versa.

The marginal rates of substitution between any two commodities are independent of the amounts of any other commodities.

Revealed Preference Analysis

If we have access to a finite data set containing price and demand observations, how do we *test* for additive separability?

Many conventional econometric approaches typically adopt functional forms and restrict observed and unobserved heterogeneity *a priori*.

The *revealed preference* approach of Afriat (1967) provides a complete characterization in terms of observables that is

- Agnostic about the specific functional shape of the preference,
- Maximally heterogeneous.

Revealed Preference Analysis

Let $\mathcal{O} = \{(p^t, x^t)\}_{t=1}^T$ be a set of observations drawn from a consumer.

Each observation consists of a price vector $p^t = (p_1^t, p_2^t, \ldots, p_\ell^t) \gg 0$ and a consumption bundle $x^t = (x_1^t, x_2^t, \ldots, x_\ell^t) \geqslant 0$.

Definition: A utility function $U : \mathbb{R}_+^\ell \to \mathbb{R}$ is said to *rationalize* the data set $\mathcal{O} = \{(p^t, x^t)\}_{t=1}^T$ if, at every observation $t = 1, 2, \ldots, T$,

$$U(x^t) \geqslant U(x) \text{ for all } x \in B^t = \{x \in \mathbb{R}_+^\ell : p^t \cdot x \leqslant p^t \cdot x^t\}.$$

Afriat (1967) answers the following question: *What are the conditions on $\mathcal{O}$ that are necessary and sufficient for it to have arisen from an agent who is maximizing a nonsatiated utility function?*

GARP

Given the data set $\mathcal{O} = \{(p^t, x^t)\}_{t=1}^T$, we denote the set of observed consumption bundles or demands by $\mathcal{D}$, i.e., $\mathcal{D} = \{x^t\}_{t=1}^T$.

For any x^t, $x^s \in \mathcal{D}$, x^t is *directly revealed (strictly) preferred* to x^s if $p^t \cdot x^s \leqslant (<)\ p^t \cdot x^t$. [Notation: $x^t \succcurlyeq^* (\succ^*)\ x^s$.]

Motivation: For an agent maximizing a nonsatiated utility function U,

$$x^t \succcurlyeq^* x^s \implies U(x^t) \geqslant U(x^s),$$

$$x^t \succ^* x^s \implies U(x^t) > U(x^s).$$

Definition: A data set $\mathcal{O} = \{(p^t, x^t)\}_{t=1}^T$ obeys the *Generalized Axiom of Revealed Preference (GARP)* if whenever there is some sequence of observations (p^{t_i}, x^{t_i}) (for $i = 1, 2, \ldots, n$) satisfying

$$x^{t_1} \succcurlyeq^* x^{t_2},\ x^{t_2} \succcurlyeq^* x^{t_3},\ \ldots,\ x^{t_{n-1}} \succcurlyeq^* x^{t_n},\ x^{t_n} \succcurlyeq^* x^{t_1},$$

then $\succcurlyeq^*$ cannot be replaced with $\succ^*$ anywhere in the chain.

GARP and Afriat's Theorem

Lemma: A data set $\mathcal{O} = \{(p^t, x^t)\}_{t=1}^T$ that is collected from an agent who is maximizing a nonsatiated utility function must obey GARP.

Afriat's Theorem: Suppose that $\mathcal{O} = \{(p^t, x^t)\}_{t=1}^T$ satisfies GARP. Then there are real numbers ϕ^t and $\lambda^t > 0$ (for all $t = 1, 2, \ldots, T$) that solve the following system of linear inequalities:

$$\phi^t \leqslant \phi^k + \lambda^k p^k \cdot (x^t - x^k) \text{ for all } k \neq t.$$

Furthermore, $\mathcal{O}$ can be rationalized by $U : \mathbb{R}_+^\ell \to \mathbb{R}$ taking the form

$$U(x) = \min_t \{\phi^t + \lambda^t p^t \cdot (x - x^t)\}.$$

Two things to notice about this result:

(1) Solving linear inequalities is computationally straightforward,

(2) U is increasing, concave, and continuous.

Afriat's Theorem

Afriat's Theorem: The following four statements on any given data set $\mathcal{O} = \{(p^t, x^t)\}_{t=1}^{T}$ are equivalent:

(1) $\mathcal{O}$ is rationalizable by a nonsatiated utility function U,

(2) $\mathcal{O}$ obeys GARP,

(3) $\mathcal{O}$ satisfies Afriat's inequalities,

(4) $\mathcal{O}$ is rationalizable by a utility function U, which is increasing, concave, and continuous.

Rationalizability by Separability

Testing for separability is typically thought to be a ‘hard’ problem, if not in concept then in computation and application.

Nonparametric tests of weak separability date back to Varian (1983) and Diewert and Parkan (1985), where the subutility functions are concave, i.e., the characterizations are not completely general.

The approach produces a bilinear test, which is computationally hard, although advances have been made, e.g., Swofford and Whitney (1987, 1988), Fleissig and Whitney (2003, 2008), and Cherchye *et al.* (2015).

More recently, Quah (2014) establishes a testing procedure for weak separability that does not impose concavity *a priori*, and which is in principle implementable on finite data sets that are not too large.

On the other hand, Echenique (2014) argues that separability is, in general, a computationally hard problem.

Rationalizability by Additive Separability

Contrary to these somewhat negative results, we argue that *additive separability* is exactly and efficiently testable.

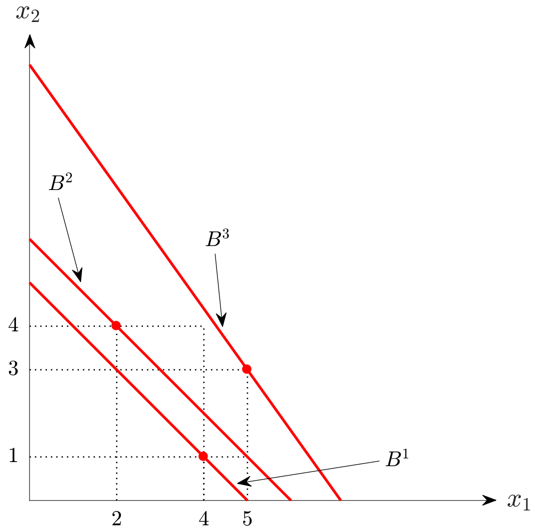
Varian (1983) and Diewert and Parkan (1985) were again the first to establish necessary and sufficient conditions for additive separability.

What is again crucial is that each subutility function is concave, which facilitates a first-order approach that produces a *joint* linear test.

Concavity implies consumption smoothing, and so any violations of the Varian (1983) and Diewert and Parkan (1985) conditions are not violations of additive separability *as such*.

The objective of this paper is to *develop* and *implement* a revealed preference test for additive separability in its purest form.

The Empirical Content of Additive Separability



$$v_1(4) + v_2(1) \geqslant v_1(2) + v_2(3), \quad v_1(2) + v_2(4) \geqslant v_1(5) + v_2(1) \quad \Longrightarrow ☹$$

A Lattice Test for Additive Separability

Definition: The data set $\mathcal{O} = \{(p^t, x^t)\}_{t=1}^T$ is *rationalizable by additive separability* if there is collection of subutility functions $\{v_i\}_{i=1}^{\ell}$, with each subutility function $v_i : \mathbb{R}_+ \to \mathbb{R}$ increasing and continuous, such that, at every observation $t = 1, 2, \ldots, T$,

$$\sum_{i=1}^{\ell} v_i(x_i^t) \geqslant \sum_{i=1}^{\ell} v_i(x_i) \text{ for all } x \in B^t,$$

where $B^t = \{x \in \mathbb{R}_+^{\ell} : p^t \cdot x \leqslant p^t \cdot x^t\}$?

We develop a *lattice test* (in the spirit of Polisson, Quah, and Renou (2017)) which has the following features:

(1) It tests for additive separability *as such*, i.e, it does not assume a convex preference *a priori*,

(2) It is applicable even when budget sets are nonconvex,

(3) It can be adapted to accommodate departures from rationality in the form cost inefficiencies (Afriat (1972, 1973); Varian (1990)).

A Lattice Test for Additive Separability

Suppose that $x^1 = (4, 1)$, $p^1 = (2, 4)$, $x^2 = (2, 3)$, $p^2 = (3, 2)$.

Define $\mathcal{X}_i = \{x_i^t : t = 1, \ldots, T\} \cup 0$, i.e., $\mathcal{X}_1 = \{0, 2, 4\}$, $\mathcal{X}_2 = \{0, 1, 3\}$.

Construct the finite lattice $\mathcal{L} = \mathcal{X}_1 \times \mathcal{X}_2$.

For additive separability, it is *necessary* that there are sets of *ordered* real numbers $\{\bar{v}_1(0), \bar{v}_1(2), \bar{v}_1(4)\}$ and $\{\bar{v}_2(0), \bar{v}_2(1), \bar{v}_2(3)\}$, such that

$$\bar{v}_1(4) + \bar{v}_2(1) \geqslant \bar{v}_1(x_1) + \bar{v}_2(x_2) \text{ for any } (x_1, x_2) \in B^1 \cap \mathcal{L},$$

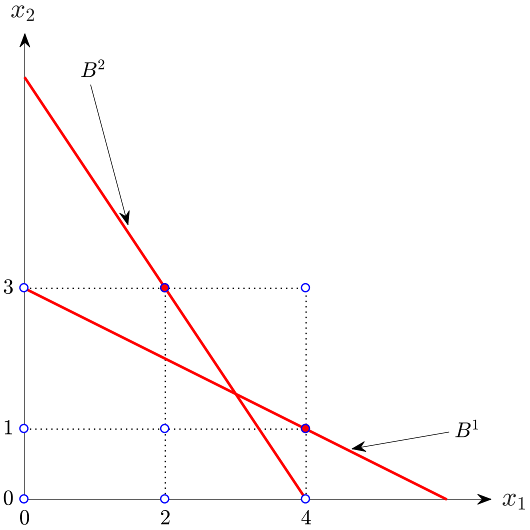
$$\bar{v}_1(2) + \bar{v}_2(3) \geqslant \bar{v}_1(x_1) + \bar{v}_2(x_2) \text{ for any } (x_1, x_2) \in B^2 \cap \mathcal{L},$$

holding strictly whenever (x_1, x_2) is in the interior of the budget set.

It is also *sufficient* for rationalizability by additive separability.

So we only need to test on a finite lattice, which is a linear test.

A Lattice Test for Additive Separability



$$\mathcal{X}_1 = \{0, 2, 4\}, \quad \mathcal{X}_2 = \{0, 1, 3\}, \quad \mathcal{L} = \mathcal{X}_1 \times \mathcal{X}_2$$

A Lattice Test for Additive Separability

Theorem: The data set $\mathcal{O} = \{(p^t, x^t)\}_{t=1}^T$ is rationalizable by additive separability if there are sets of ordered real numbers $\{\bar{v}_i(x_i)\}_{i=1}^{\ell}$, such that, at every observation $t = 1, 2, \ldots, T$,

$$\sum_{i=1}^{\ell} \bar{v}_i(x_i^t) \geqslant \sum_{i=1}^{\ell} \bar{v}_i(x_i) \text{ for all } x \in B^t \cap \mathcal{L},$$

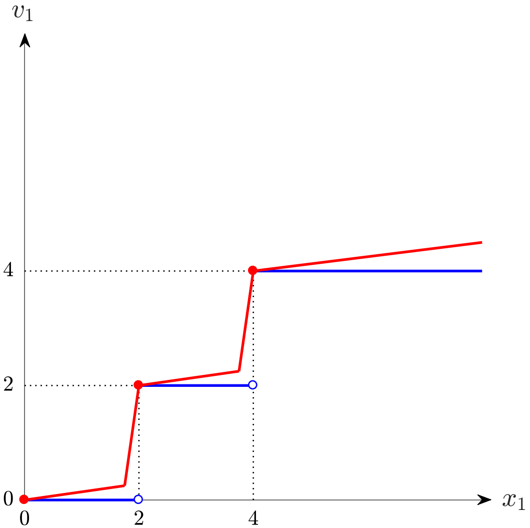
$$\sum_{i=1}^{\ell} \bar{v}_i(x_i^t) > \sum_{i=1}^{\ell} \bar{v}_i(x_i) \text{ for all } x \in (B^t \backslash \partial B^t) \cap \mathcal{L}.$$

Intuition: First we replace each $\bar{v}_i$ with a step function $\hat{v}_i : \mathbb{R}_+ \to \mathbb{R}$ so that $\hat{v}_i(y_i) = \bar{v}_i(y_i)$ for all $y_i \in \mathcal{X}_i$, with $\hat{v}_i$ constant in between the values of $\mathcal{X}_i$. Clearly, the data are rationalizable in the sense that

$$\sum_{i=1}^{\ell} \hat{v}_i(x_i^t) \geqslant \sum_{i=1}^{\ell} \hat{v}_i(x_i) \text{ for all } x \in B^t.$$

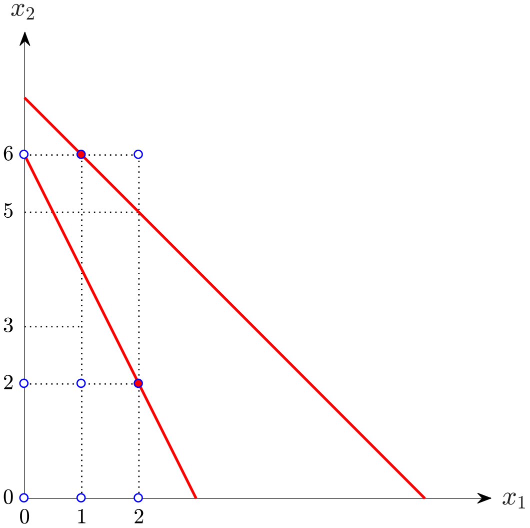
The only problem is that each $\hat{v}_i$ is neither increasing nor continuous. But it is possible to find some function v_i, arbitrarily close to $\hat{v}_i$, that is increasing and continuous ...

A Lattice Test for Additive Separability



$$\mathcal{X}_1 = \{0, 2, 4\}, \ \bar{v}_1(0) = 0, \ \bar{v}_1(2) = 2, \ \bar{v}_1(4) = 4$$

A Violation of Concave Additive Separability



$$v_1(2) + v_2(2) > v_1(1) + v_2(3), \quad v_2(6) - v_2(3) \leqslant v_2(5) - v_2(2) \quad \Longrightarrow ☹$$

Critical Cost Efficiency Index

In order to accommodate departures from rationality, we adopt an approach first suggested by Afriat (1972, 1973) and Varian (1990).

The data set $\mathcal{O} = \{(p^t, x^t)\}_{t=1}^T$ is rationalizable by some family $\mathbf{U}$ if there is a utility function $U : \mathbb{R}^\ell_+ \to \mathbb{R}$ belonging to $\mathbf{U}$ such that

$$U(x^t) \geqslant U(x) \text{ for all } x \in B^t = \{x \in \mathbb{R}^\ell_+ : p^t \cdot x \leqslant p^t \cdot x^t\}.$$

If no function in $\mathbf{U}$ rationalizes $\mathcal{O}$, we can make the requirement less stringent by shrinking all budget sets in $\mathcal{O}$ by a factor $e \in [0, 1)$.

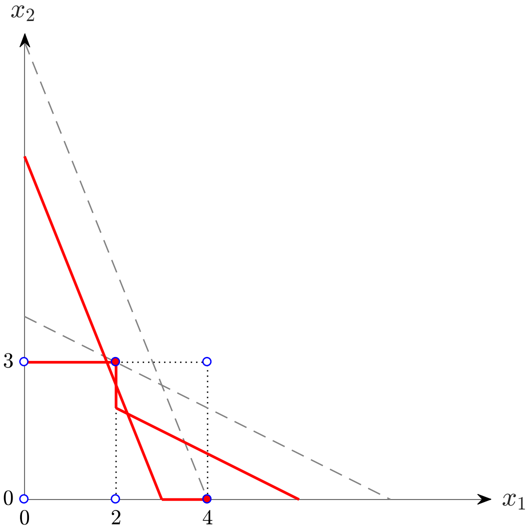
We find U in $\mathbf{U}$ such that $U(x^t) \geqslant U(x)$ for all $x \in B^t(e)$, where

$$B^t(e) = \{x \in \mathbb{R}^S_+ : x \leqslant x^t\} \cup \{x \in \mathbb{R}^\ell_+ : p^t \cdot x \leqslant e\, p^t \cdot x^t\}.$$

The largest e at which a data set passes the test is known as the *critical cost efficiency index (CCEI)* associated with $\mathcal{O}$ and $\mathbf{U}$.

Notice that $B^t(e)$ is *not* a convex set.

Critical Cost Efficiency Index



$$e = 0.75, \ \ (2,3) \succ_e^* (4,0), \ \ (4,0) \not\succeq_e^* (2,3)$$

Implementation on Household Food Consumption

We apply our tests to a representative sample of British households from the Kantar Worldpanel, which is a rolling panel that records (via a barcode scanner) all food purchases which enter the home.

Data from 2005–2012 are aggregated to a household-year-month level.

The sample contains 4,027 households, and 50 observations on each.

We focus our analysis on 4 aggregate food groups: fruit, vegetables, red meat, and poultry/fish.

Unit prices are constructed at the household-year-month level, and any missing prices (which are due to zero purchases) are imputed.

Market Summary Statistics

	Budget Shares	Unit Prices
Fruit	8.01 (6.05)	1.55 (0.87)
Vegetables	9.54 (5.25)	1.43 (0.75)
Red Meat	13.79 (7.63)	5.03 (2.03)
Poultry/Fish	6.84 (5.25)	5.16 (2.54)

Table: Market Summary Statistics

Fruit, vegetables, red meat, and poultry/fish account for roughly 38 percent of a household's monthly spending on food.

The largest share of the budget goes towards red meat, followed by vegetables, then fruit, and finally poultry/fish; furthermore, red meat and poultry/fish are more expensive than fruit and vegetables.

Rationalizability Results

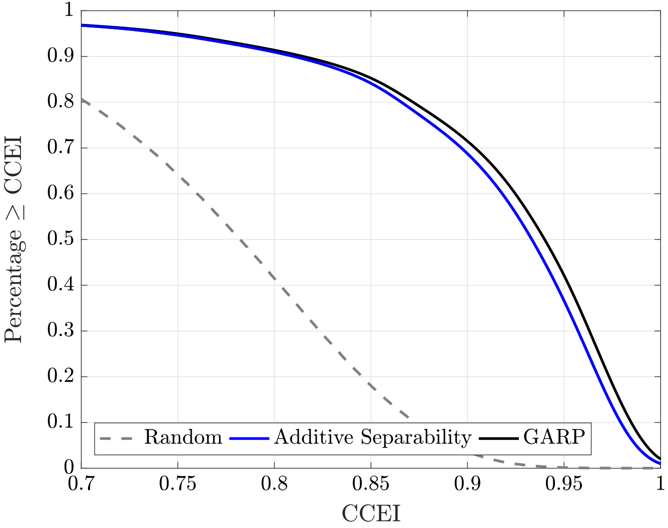
We assume weak separability between subgroups {fruit, vegetables} and {red meat, poultry/fish}.

These assumptions allow us to focus our attention *within* each group, i.e., on the interaction *between* fruit and vegetables and *between* red meat and poultry/fish.

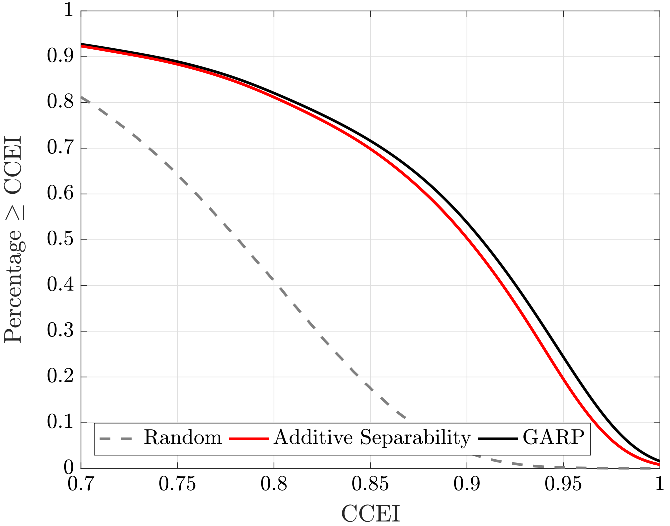
	Utility Maximization		Additive Separability	
	CCEI	95% CI	CCEI	95% CI
Fruit, Vegetables	0.9143 (0.0013)	$[0.9117, 0.9169]$	0.9084 (0.0013)	$[0.9058, 0.9110]$
Red Meat, Poultry/Fish	0.8765 (0.0017)	$[0.8732, 0.8798]$	0.8701 (0.0017)	$[0.8668, 0.8733]$

Table: Rationalizability Results

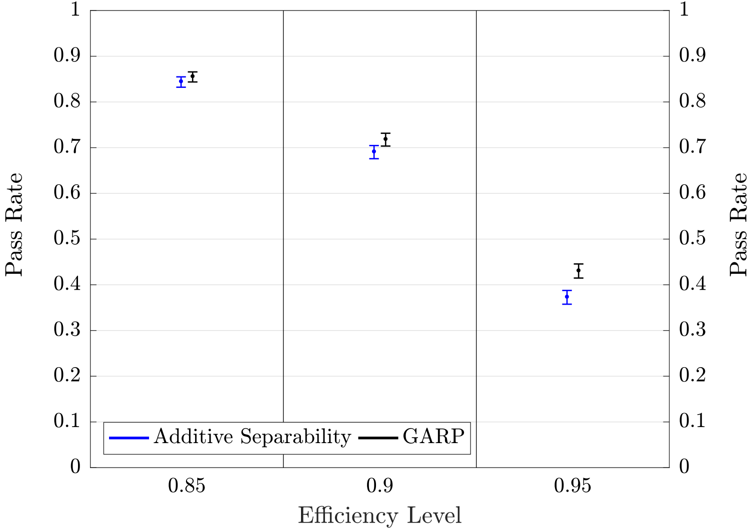
CCEI Distributions (Fruit, Vegetables)



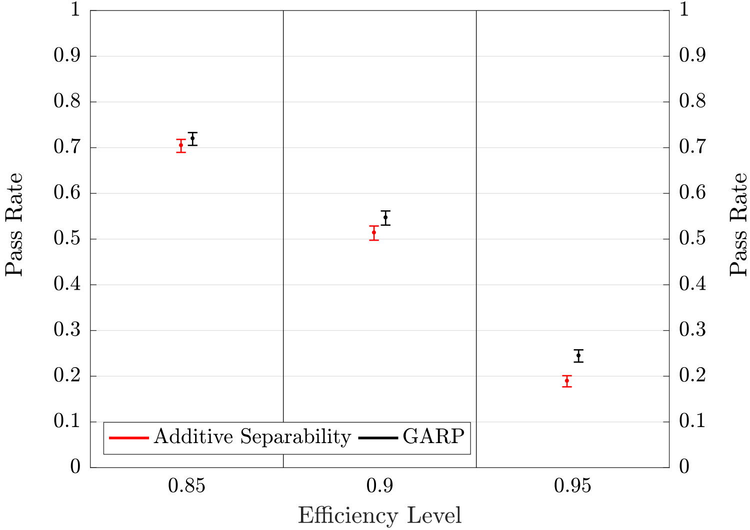
CCEI Distributions (Red Meat, Poultry/Fish)



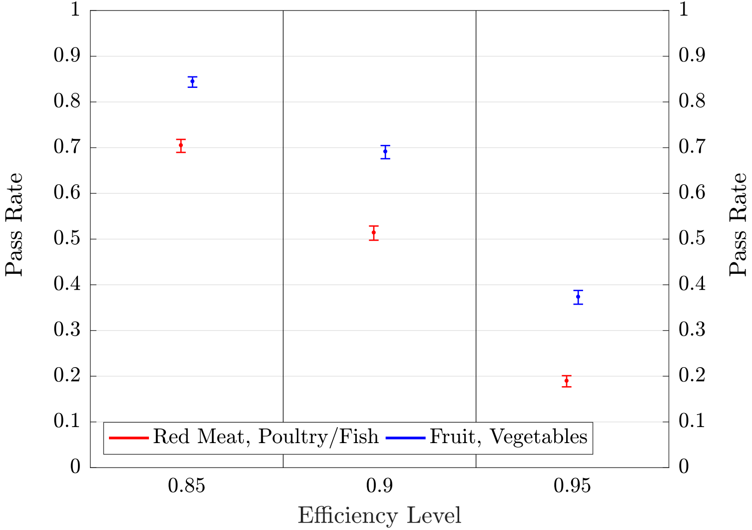
Pass Rates (Fruit, Vegetables)



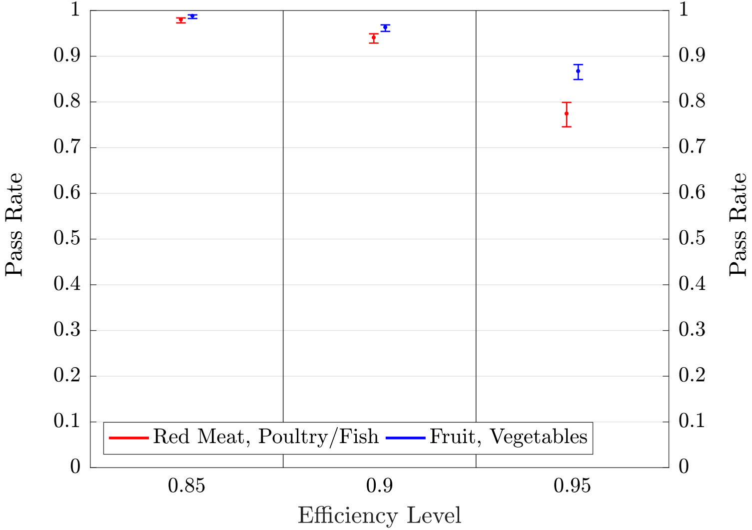
Pass Rates (Red Meat, Poultry/Fish)



Pass Rates for Additive Separability



Conditional Pass Rates for Additive Separability



Bronars Power

Rationalizability, exact or approximate, only measures whether a given data set is consistent with a particular model; but different models can be more or less observationally stringent.

Bronars (1987) proposed to measure the *power* of a model as the probability of a random (uniform) consumer failing the test.

	Utility Maximization		
	$e = 0.85$	$e = 0.90$	$e = 0.95$
Fruit, Vegetables	0.82	0.96	1.00
Red Meat, Poultry/Fish	0.82	0.96	1.00

Table: Power Results

Power of Additive Separability

	Additive Separability		
	$e = 0.85$	$e = 0.90$	$e = 0.95$
Fruit, Vegetables	0.85	0.98	1.00
Red Meat, Poultry/Fish	0.86	0.98	1.00

Table: Power Results

	Additive Separability		
	$e = 0.85$	$e = 0.90$	$e = 0.95$
Fruit, Vegetables	0.43	0.76	0.98
Red Meat, Poultry/Fish	0.45	0.79	0.99

Table: Conditional Power Results

Selten Index of Predictive Success

Selten (1991) proposed an *index of predictive success* to measure the degree to which a model is able to explain a given data set:

$$\text{Hit Rate} - \mu(\text{Model-Consistent Outcomes}).$$

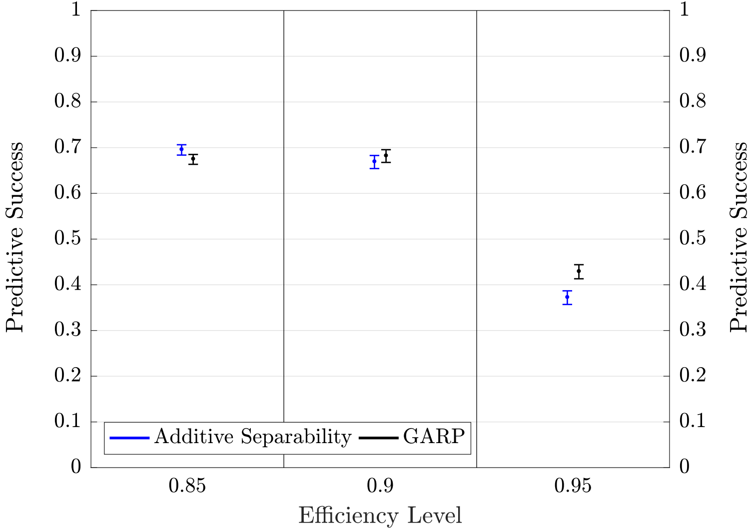
The Hit Rate is the observed frequency of model-consistent outcomes.

The μ(Model-Consistent Outcomes) is the probability of a random outcome being model-consistent; smaller μ means greater *precision.*

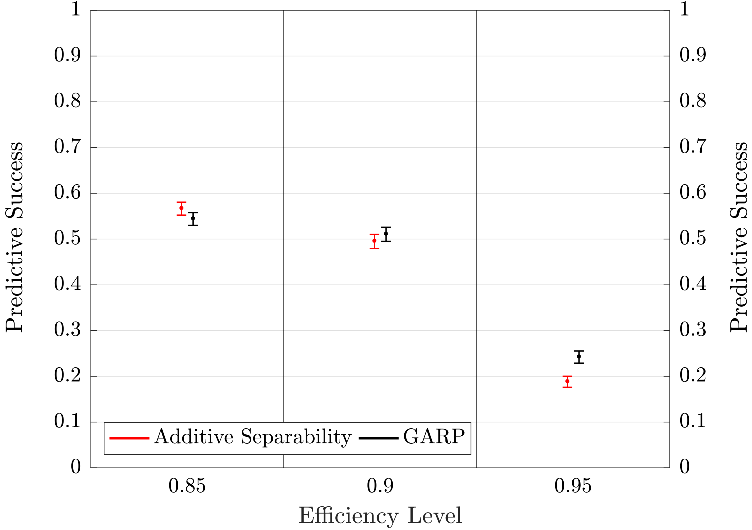
A good model has a high hit rate and high precision (small μ); a poor model has a low hit rate and low precision (large μ).

The Selten index takes values in $[-1, 1]$; any model with a Selten index above 0 can be considered to have some predictive success.

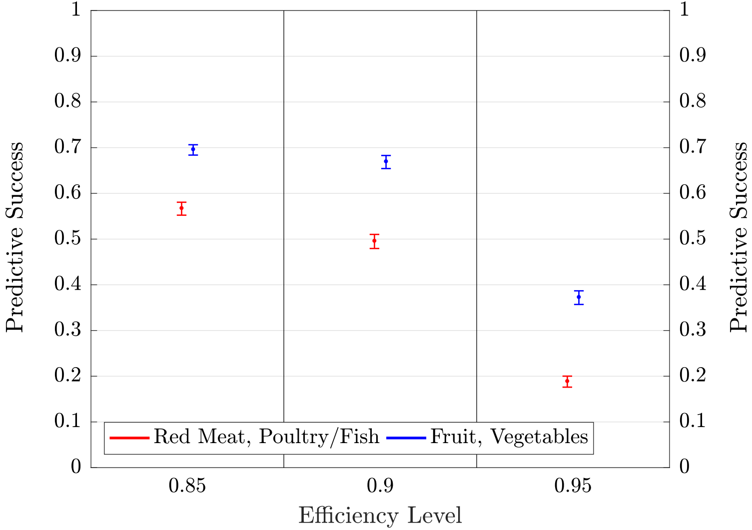
Predictive Success (Fruit, Vegetables)



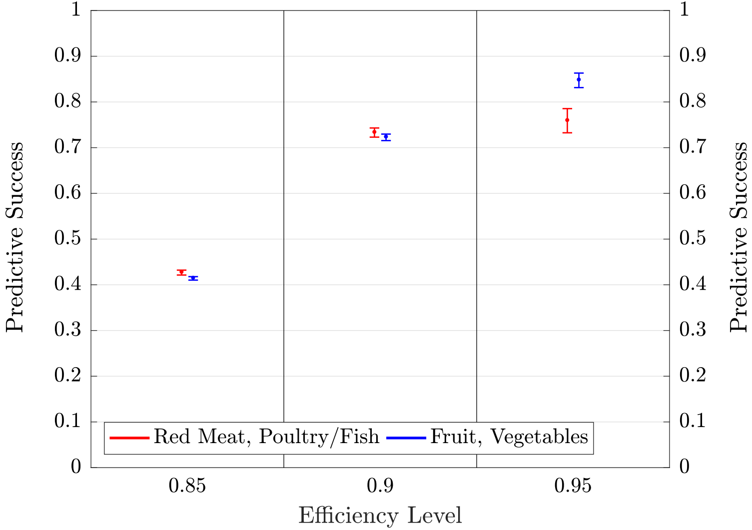
Predictive Success (Red Meat, Poultry/Fish)



Predictive Success for Additive Separability



Conditional Predictive Success for Additive Separability



Conclusions

The lattice approach to testing for additive separability:

(1) Avoids ancillary functional form assumptions,

(2) Allows for maximal heterogeneity,

(3) Accommodates departures from rationality,

(4) Facilitates comparisons across groups.

The empirical finding is that additive separability has considerable success in explaining consumption choices within both subgroups.

The result is more pronounced for {fruit, vegetables} than for {red meat, poultry/fish}, suggesting greater demand independence within the former subgroup.